STEADY STATES WITH UNBOUNDED MASS OF THE KELLER-SEGEL SYSTEM

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ABSTRACT. We consider the boundary value problem

$$\begin{cases} -\Delta u + u = \lambda e^u, & \text{in } B_{r_0} \\ \partial_\nu u = 0 & \text{on } \partial B_{r_0} \end{cases}$$

where B_{r_0} is the ball of radius r_0 in $\mathbb{R}^N$, $N \geq 2$, $\lambda > 0$ and ν is the outer normal derivative at ∂B_{r_0}. This problem is equivalent to the stationary Keller-Segel system from chemotaxis.
We show the existence of a solution concentrating at the boundary of the ball as λ goes to zero.

1. Introduction

We consider a system of partial differential equations modelling chemotaxis. Chemotaxis is a phenomenon of the direct movement of cells in response to the gradient of a chemical, which explains the aggregation of cells which move towards high concentration of a chemical secreted by themselves. The basic model was introduced by Keller and Segel in [15] and a simplified form of it reads

$$\begin{cases} v_t = \Delta v - \nabla(v\nabla u) & \text{in } \Omega \\ \tau u_t = \Delta u - u + v & \text{in } \Omega \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & \end{cases} \tag{1.1}$$

where $u = u(x,t) \geq 0$ and $v = v(x,t) \geq 0$ are the concentration of the species and that of chemical. Here Ω is a bounded smooth domain in $\mathbb{R}^N$ and $N \geq 2$. The cases $N = 2$ or $N = 3$ are of particular interest. In (1.1) ν denotes the unit outward vector normal at $\partial\Omega$ and τ is a positive constant.

After the seminal works by Nanjudiah [20] and Childress and Percus [4] many contributions have been made to the understanding of different analytical aspects of this system and its variations. We refer the reader for instance to [2, 5, 6, 11, 12, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27].

In this paper, we study steady states of (1.1), namely solutions to the system

$$\begin{cases} \Delta v - \nabla(v\nabla u) = 0 & \text{in } \Omega \\ \Delta u - u + v = 0 & \text{in } \Omega \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

As point out in [18], stationary solutions to the Keller-Segel system are of basic importance for the understanding of the global dynamic of the system.

This problem was first studied by Schaaf in [21] in the one dimensional case. In the higher dimensional case Biler in [1] proved the existence of nontrivial radially symmetric solution to (1.2) when Ω is a ball. In the general two dimensional case, Wang and Wei in [28] and Senba and Suzuki in [22] proved that for any $\mu \in \left(0, \frac{1}{|\Omega|} + \mu_1\right) \setminus \{4\pi m \ :\ m \geq 1\}$ problem (1.2) has a non constant solution such that $\int\limits_\Omega v(x)dx = \mu|\Omega|$. Here μ_1 is the first eigenvalue of $-\Delta$ with Neumann boundary conditions. Del Pino and Wei in [3] reduced system (1.2) to a scalar equation. Indeed, it is easy

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to check that (u,v) solves system (1.2) if and only if $v=\lambda e^u$ for some positive constant λ and u solves the equation

$$\begin{cases} -\Delta u+u=\lambda e^u & \text{in } \Omega \\ \partial_\nu u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Using this point of view, they proved that for any integers k and ℓ there exists a family of solutions (u_λ,v_λ) to the system (1.2) such that v_λ exhibits k Dirac measures inside the domain and ℓ Dirac measures on the boundary of the domain as $\lambda\to 0$, i.e.

$$v_\lambda \rightharpoonup \sum_{i=1}^{k} 8\pi\delta_{\xi_i}+\sum_{i=1}^{\ell} 4\pi\delta_{\eta_i} \quad \text{as } \lambda\to 0,$$

where $\xi_1,\dots,\xi_k\in\Omega$ and $\eta_1,\dots,\eta_\ell\in\partial\Omega$. In particular, the solution has bounded mass, i.e.

$$\lim_{\lambda\to 0}\int_\Omega v_\lambda(x)dx=\lim_{\lambda\to 0}\int_\Omega \lambda e^{u_\lambda(x)}dx=4\pi(2k+\ell).$$

In particular, their argument allows to find a radial solution to the system (1.2) when Ω is a ball in $\mathbb{R}^2$, which exhibits a Dirac measure at the center of the ball with mass 8π when λ goes to zero.

In the present paper, we find a new radial solution to the system (1.2) when Ω is a ball in $\mathbb{R}^N$, $N\ge 2$, with unbounded mass. Our main result reads as follows.

Theorem 1.1. *Let $\Omega=B(0,r_0)\subset\mathbb{R}^N$, $N\ge 2$, be the ball centered at the origin with radius r_0. There exists λ_0 such that for any $\lambda\in(0,\lambda_0)$, the problem* (1.3) *has a radial solution (u_λ,v_λ) such that as $\lambda\to 0$*

$$\lim_{\lambda\to 0}\int_\Omega v_\lambda(x)dx=\lim_{\lambda\to 0}\int_\Omega \lambda e^{u_\lambda(x)}dx=+\infty. \tag{1.4}$$

Moreover, for a suitable choice of positive numbers ϵ_λ (see (2.11)*) with $\epsilon_\lambda\to 0$ as $\lambda\to 0$, we have*

$$\lim_{\lambda\to 0}\epsilon_\lambda u_\lambda=\frac{\sqrt{2}}{\mathcal{U}'(r_0)}\mathcal{U} \qquad C^0\text{−uniformly on compact sets of } \Omega. \tag{1.5}$$

Here $\mathcal{U}$ is the positive radial solution to the problem (see also Lemma 2.1)

$$\begin{cases} -\Delta\mathcal{U}+\mathcal{U}=0 & \text{in } B(0,r_0) \\ \mathcal{U}=1 & \text{on } \partial B(0,r_0). \end{cases} \tag{1.6}$$

To find this solution, we use a fixed point argument. More precisely, we look for a solution to equation (1.3) as $u_\lambda=\bar u_\lambda+\phi_\lambda$, where the leading term $\bar u_\lambda$ has to be accurately defined. Once one has a good approximating solution $\bar u_\lambda$, a simple contraction mapping argument leads to find the higher order term ϕ_λ.

The difficulty in the construction of the approximated solution $\bar u_\lambda$ is due to the fact that $\bar u_\lambda$ shares the behavior of $\mathcal{U}$ (which solves (1.6)) in the inner part of the ball and the behavior of the function w_ϵ (see (1.7)) near the boundary of the ball. Here

$$w_\epsilon(r)=\ln\frac{4}{\epsilon^2}\frac{e^{\sqrt{2}\frac{r-r_0}{\epsilon}}}{(1+e^{\sqrt{2}\frac{r-r_0}{\epsilon}})^2},\ r\in\mathbb{R},\ \epsilon>0. \tag{1.7}$$

solve the one dimensional limit problem

$$-w''=e^w \qquad \text{in } \mathbb{R}. \tag{1.8}$$

In particular, we have to spend a lot of effort to glue the two functions up to the third order (see (2.11), (2.12) and (3.17)) in a neighborhood of the boundary (see Lemma 4.1).

It is important to remark about the analogy existing between our result and some recent results obtained by Grossi in [7, 8, 9, 10]. In particular, Grossi and Gladiali in [10] studied the asymptotic behavior as λ goes to zero of the radial solution z_λ to the Dirichlet problem

$$\begin{cases} -\Delta z = \lambda e^z & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

when $\Omega = \{x \in \mathbb{R}^n \ : \ a < |x| < b\}$ is the annulus in $\mathbb{R}^n$. In particular, they proved that for a suitable choice of positive numbers δ_λ with $\delta_\lambda \to 0$ as $\lambda \to 0$, z_λ satisfies

$$\lim_{\lambda\to 0} \delta_\lambda z_\lambda(r) = 2\sqrt{2}G(r, r^*) \qquad C^0\text{−uniformly on compact sets of } (a, b).$$

where $G(\cdot, r^*)$ is the Green's function of the radial Laplacian with Dirichlet boundary condition and r^* is suitable choose in (a, b). Moreover, a suitable scaling of z_λ in a neighborhood of r^* converges (as λ goes to zero) at a solution of the one dimensional limit problem (1.8).

The paper is organized as follows. The definition of $\bar{u}_\lambda$ is given in Section 2, while the construction of a good approximation near the boundary of the ball is carried out in Section 3. In Section 4 we estimate the error term and in Section 5 we apply the contraction mapping argument.

2. The approximated solution

We look for a radial solution to the problem (1.3), so we are leading to consider the ODE problem

$$\begin{cases} -u'' - \dfrac{N-1}{r}u' + u = \lambda e^u & \text{in } (0, r_0) \\ u'(r_0) = 0 \\ u'(0) = 0. \end{cases} \tag{2.9}$$

We will construct a solution to (2.9) as $\bar{u}_\lambda + \phi_\lambda$ where the leading term $\bar{u}_\lambda$ is defined as

$$\bar{u}_\lambda(r) := \begin{cases} u_1(r) & \text{in } (r_0 - \delta, r_0) \\ u_2(r) & \text{in } [r_0 - 2\delta, r_0 - \delta] \\ u_3(r) & \text{in } (0, r_0 - 2\delta) \end{cases} \tag{2.10}$$

and u_1, u_2 and u_3 are defined as follows.

Basic cells in the construction of the approximate solution u_1 near r_0 are the functions w_ϵ defined in (1.7). The rate of the concentration parameter $\epsilon := \epsilon_\lambda$ with respect to λ is deduced by the relation

$$\lambda = \frac{4}{\epsilon_\lambda^2} e^{-\left(\frac{a_1}{\epsilon_\lambda} + a_2 + a_3\epsilon_\lambda\right)}, \quad \text{i.e.} \quad \ln\frac{4}{\epsilon_\lambda^2} - \ln\lambda = \frac{a_1}{\epsilon_\lambda} + a_2 + a_3\epsilon_\lambda \tag{2.11}$$

where a_1, a_2 and a_3 are positive constants given in (4.39).
The right expression of u_1 is given in (3.17). The construction of u_1 is quite involved and it will be carried out in Section 3.

The approximate solution u_3 far away from r_0 is build from the function $\mathcal{U}$ which solves (1.6) and whose properties are stated in Lemma 2.1.

More precisely,

$$u_3(r) = \left(\frac{A_1}{\epsilon_\lambda} + A_2 + A_3\epsilon_\lambda\right)\mathcal{U}(r) \tag{2.12}$$

where A_1, A_2 and A_3 are positive constants given in (4.39).

Finally, the approximate solution u_2 in the interspace is simply given by

$$u_2(r) := \chi(r)u_1(r) + (1 - \chi(r))u_3(r) \tag{2.13}$$

where $\chi \in C^2([0,r_0])$ is a cut-off such that

$$\chi \equiv 1 \text{ in } (r_0-\delta, r_0),\ \chi \equiv 0 \text{ in } (0, r_0-2\delta),\ |\chi(r)| \le 1,\ |\chi'(r)| \le \frac{c}{\delta},\ |\chi''(r)| \le \frac{c}{\delta^2}. \tag{2.14}$$

where the size of the interface $\delta := \delta_\lambda$ is going to zero with respect to ϵ (or equivalently with respect to λ) as

$$\delta_\lambda = \epsilon_\lambda^\eta, \quad \eta \in \left(\frac{2}{3}, 1\right). \tag{2.15}$$

The choice of η will be made so that Lemma (4.2) holds.

It is important to point out that u_2 is a good approximation of the solution in the interspace, if u_1 and u_3 perfectly glue in a left neighborhood of r_0. That implies that we need to go into a third order expansion in u_1 (see (3.17)) and in u_3 (see (2.12)) and also motivates the rate of ϵ_λ made in (2.11) and the choice of the constants A_1, A_2, A_3 and a_1, a_2, a_3 made in Lemma 4.1.

Lemma 2.1. *There exists a unique solution to the problem*

$$\begin{cases} -\mathcal{U}'' - \dfrac{N-1}{r}\mathcal{U}' + \mathcal{U} = 0 & \text{in } (0, r_0) \\ \\ \mathcal{U}'(0) = 0,\ \mathcal{U}(r_0) = 1. \end{cases} \tag{2.16}$$

Moreover

$$0 \le \mathcal{U}(r) \le 1 \text{ and } \mathcal{U}'(r) > 0 \text{ for any } r \in (0, r_0].$$

Proof The existence and uniqueness of the solution are standard. By the maximum principle we deduce that $\mathcal{U} \le 1$ in $(0, r_0]$.

If $r^* \in (0, r_0)$ is a minimum point of $\mathcal{U}$ with $\mathcal{U}(r^*) < 0$, by (2.16) we deduce that $\mathcal{U}''(r^*) = \mathcal{U}(r^*) < 0$ which is not possible. So $\mathcal{U} \ge 0$ in $(0, r_0]$.

Finally, we integrate (2.16) and we get

$$r^{N-1}\mathcal{U}'(r) = \int_0^r t^{N-1}\mathcal{U}(t)dt \ge 0 \text{ for any } r \in (0, r_0],$$

which implies $\mathcal{U}' > 0$ in $(0, r_0]$. □

3. The approximation near the boundary

The function $w_\epsilon - \ln\lambda$ is not a good approximation for our solution near r_0. We will build some additional correction terms which improve the approximation near r_0. More precisely, we define the approximation near the point r_0. We define

$$u_1(r) = \underbrace{w_\epsilon(r) - \ln\lambda + \alpha_\epsilon(r)}_{1^{st} \text{ order approx.}} + \underbrace{v_\epsilon(r) + \beta_\epsilon(r)}_{2^{nd} \text{ order approx.}} + \underbrace{z_\epsilon(r)}_{3^{rd} \text{ order approx.}} \tag{3.17}$$

where α_ϵ is defined in Lemma 3.1, v_ϵ is defined in Lemma 3.2, β_ϵ is defined in Lemma 3.4 and z_ϵ is defined in Lemma 3.5.

The first term we have to add is a sort of projection of the function w_ϵ, namely the function α_ϵ given in the next lemma.

Lemma 3.1. (i) *The Cauchy problem*

$$\begin{cases} -\alpha''_{\epsilon,N} - \dfrac{N-1}{r}\alpha'_{\epsilon,N} = \dfrac{N-1}{r}w'_\epsilon(r) - w_\epsilon(r) + \ln\lambda & \text{in } (0,r_0) \\ \alpha_\epsilon(r_0) = \alpha'_\epsilon(r_0) = 0. \end{cases} \tag{3.18}$$

has the solution

$$\alpha_\epsilon(r) := -\int_{r_0}^{r} \frac{1}{t^{N-1}} \int_{r_0}^{t} \tau^{N-1}\left[\frac{N-1}{\tau}w'_\epsilon(\tau) - w_\epsilon(\tau) + \ln\lambda\right] d\tau\ dt.$$

(ii) *The following expansion holds*

$$\alpha_\epsilon(\epsilon s + r_0) = \epsilon\alpha_1(s) + \epsilon^2\alpha_2(s) + O\left(\epsilon^3 s^4\right) \tag{3.19}$$

where

$$\alpha_1(s) := -\frac{N-1}{r_0}\int_0^s w(\sigma)\ d\sigma + \frac{1}{2}a_1 s^2 \tag{3.20}$$

and

$$\begin{aligned} \alpha_2(s) \quad := \quad & \int_0^s \int_0^\sigma [w(\rho) - \ln 4]\ d\rho\ d\sigma + \frac{(N-1)(N-2)}{r_0^2}\int_0^s \int_0^\sigma w(\rho)\, d\rho\ d\sigma \\ & + \frac{(N-1)}{r_0^2}\int_0^s \sigma w(\sigma)\ d\sigma - \frac{N-1}{6r_0}a_1 s^3 + \frac{1}{2}a_2 s^2 \end{aligned} \tag{3.21}$$

(iii) *For any $r \in (0, r_0 - \delta)$*

$$\begin{aligned} \alpha_\epsilon(r) \quad = \quad & -\frac{(N-1)\ln 4}{r_0}(r - r_0) + \left[\frac{(N-1)^2\ln 4}{r_0^2} - \frac{\sqrt{2}(N-1)}{\epsilon r_0} + \ln\frac{4}{\epsilon^2} - \ln\lambda\right]\frac{(r-r_0)^2}{2} \\ & + \left[\frac{N(N-1)\sqrt{2}}{\epsilon r_0^2} + \frac{\sqrt{2}}{\epsilon} - \frac{N-1}{r_0}\left(\ln\frac{4}{\epsilon^2} - \ln\lambda\right)\right]\frac{(r-r_0)^3}{6} \\ & + O\left(\frac{(r-r_0)^4}{\epsilon}\right) + O\left((r-r_0)^3\right) \end{aligned} \tag{3.22}$$

Proof

Proof of (i). It is just a straightforward computation.

Proof of (ii).
We get (setting $t = \epsilon\sigma + r_0$ and $\tau = \epsilon\rho + r_0$)

$$\begin{aligned} \alpha_\epsilon(\epsilon s + r_0) \quad & = -\epsilon^2 \int_0^s \frac{1}{(\epsilon\sigma + r_0)^{N-1}} \int_0^\sigma (\epsilon\rho + r_0)^{N-1}\left[\frac{N-1}{\epsilon\rho + r_0}\frac{1}{\epsilon}w'(\rho) - [w(\rho) - \ln 4] + \ln\lambda - \ln\frac{4}{\epsilon^2}\right] d\sigma\ d\rho \\ & = -\epsilon^2 \int_0^s \left(\frac{1}{r_0^{N-1}} - \frac{N-1}{r_0^N}\epsilon\sigma\right) \int_0^\sigma \left(r_0^{N-1} + (N-1)r_0^{N-2}\epsilon\rho\right) \times \\ & \quad \times \left[(N-1)\left(\frac{1}{r_0} - \frac{1}{r_0^2}\epsilon\rho\right)\frac{1}{\epsilon}w'(\rho) - [w(\rho) - \ln 4] + \ln\lambda - \ln\frac{4}{\epsilon^2}\right] d\sigma\ d\rho \\ & + O\left(\epsilon^3 s^4\right) \end{aligned}$$

Here we used that

$$w_\epsilon(r) = \ln\frac{4}{\epsilon^2} + w\left(\frac{r - r_0}{\epsilon}\right) - \ln 4 \text{ and } w'_\epsilon(r) = \frac{1}{\epsilon}w'\left(\frac{r - r_0}{\epsilon}\right).$$

The claim follows by (2.11).

Proof of (iii). Set $\bar w_\epsilon(r) := w_\epsilon(r) - \ln\frac{1}{\epsilon^2}$.
We have

$$\begin{aligned}\alpha_\epsilon(r) &= -\int_{r_0}^{r}\frac{1}{t^{N-1}}\int_{r_0}^{t}\tau^{N-1}\left[\frac{N-1}{\tau}w'_\epsilon(\tau) - w_\epsilon(\tau) + \ln\lambda\right]d\tau\ dt\\ &= -\int_{r_0}^{r}\frac{1}{t^{N-1}}\int_{r_0}^{t}\tau^{N-1}\left[\frac{N-1}{\tau}\bar w'_\epsilon(\tau) - \bar w_\epsilon(\tau) + \left(\ln\lambda - \ln\frac{1}{\epsilon^2}\right)\right]d\tau\ dt\\ &= -(N-1)\int_{r_0}^{r}\frac{\bar w_\epsilon(t)}{t}dt + \int_{r_0}^{r}\frac{1}{t^{N-1}}\int_{r_0}^{t}\left[(N-1)(N-2)\tau^{N-3} + \tau^{N-1}\right]\bar w_\epsilon(\tau)d\tau\ dt\\ &+ \left(\ln\lambda - \ln\frac{1}{\epsilon^2}\right)\begin{cases}\frac{1}{2N}(r_0^2 - r^2) + \frac{r_0^2}{2}\ln\frac{r}{r_0} \text{ if } N = 2\\ \frac{1}{2N}(r_0^2 - r^2) + \frac{r_0^N}{N(N-2)}\left(\frac{1}{r_0^{N-2}} - \frac{1}{r^{N-2}}\right) \text{ if } N \ge 3.\end{cases}\end{aligned}$$

Now we observe that in $[r_0 - 2\delta, r_0 - \delta]$ we have

$$\ln\frac{r}{r_0} = \ln\left(1 + \frac{r - r_0}{r_0}\right) = \frac{r - r_0}{r_0} - \frac{(r-r_0)^2}{2r_0^2} + \frac{(r-r_0)^3}{3r_0^3} + O\left((r-r_0)^4\right) \tag{3.23}$$

$$\frac{1}{r^{N-2}} = \frac{1}{r_0^{N-2}} - \frac{(N-2)}{r_0^{N-1}}(r-r_0) + \frac{(N-2)(N-1)}{r_0^N}\frac{(r-r_0)^2}{2} - \frac{N(N-1)(N-2)}{r_0^{N+1}}\frac{(r-r_0)^3}{6} + O((r-r_0)^4) \tag{3.24}$$

and also

$$\bar w_\epsilon(s) = \ln 4 + \frac{\sqrt{2}}{\epsilon}(s - r_0) + O\left(e^{-\frac{|s-r_0|}{\epsilon}}\right). \tag{3.25}$$

A tedious but straightforward computation proves our claim.

□

The function $w_\epsilon(r) - \ln\lambda + \alpha_\epsilon(r)$ is yet a bad approximation of the solution near the boundary point r_0. We have to add a correction term v_ϵ given in next lemma, which solves a linear problem and *kills* the $\epsilon-$order term in (3.19).

Lemma 3.2. (i) *There exists a solution v of the linear problem (see* (3.20)*)*

$$-v'' - e^w v = e^w\alpha_1 \quad \textit{in } \mathbb{R} \tag{3.26}$$

such that

$$v(s) = \nu_1 s + \nu_2 + O(e^s) \quad \textit{and} \quad v'(s) = \nu_1 + O(e^s) \quad \textit{as } s \to -\infty$$

where $\nu_2 \in \mathbb{R}$ and

$$\nu_1 := -\frac{2(N-1)}{r_0}(1 - \ln 2) + a_1\sqrt{2}\ln 2 \tag{3.27}$$

(ii) *In particular, the function $v_\epsilon(r) := \epsilon v\left(\frac{r-r_0}{\epsilon}\right)$ is a solution of the linear problem*

$$-v''_\epsilon - e^{w_\epsilon}v_\epsilon = \epsilon e^{w_\epsilon(r)}\alpha_1\left(\frac{r-r_0}{\epsilon}\right) \quad \textit{in } \mathbb{R} \tag{3.28}$$

such that if $r \in [0, r_0 - \delta]$ it satisfies

$$v_\epsilon(r) = \nu_1(r - r_0) + \nu_2\epsilon + O(\epsilon e^{-\frac{|r-r_0|}{\epsilon}}) \quad \textit{and} \quad v'_\epsilon(r) = \nu_1 O(e^{-\frac{|r-r_0|}{\epsilon}}) \quad \textit{as } \epsilon \to 0. \tag{3.29}$$

Proof The result immediately follows by Lemma 3.3. In our case is

$$\nu_1 := \frac{1}{\sqrt{2}} \int_{-\infty}^{0} \left(-\frac{N-1}{r_0} \int_0^r w(y)\, dy + a_1 \frac{r^2}{2} \right) w'(r) e^w(r) dr$$

and a straightforward computation proves (3.27). □

Lemma 3.3. *[9], Lemma 4.1] Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. The function*

$$Y(t) = w'(t) \int_0^t \frac{1}{w'(s)^2} \left(\int_s^0 h(z) w'(z) e^w\, dz \right) ds \tag{3.30}$$

is a solution to

$$-Y'' - e^w Y = h e^w \qquad \textit{in } \mathbb{R} \tag{3.31}$$

Moreover, it satifies

$$Y(t) = \frac{t}{\sqrt{2}} \int_{-\infty}^0 h(r) w'(r) e^w\, dr - \int_{-\infty}^0 \left(\frac{2}{1 - e^{\sqrt{2}s}} + \frac{s}{\sqrt{2}} \right) h(s) w'(s) e^w\, ds + O\left(e^t\right),$$
$$Y'(t) = \frac{1}{\sqrt{2}} \int_{-\infty}^0 h(r) w'(r) e^w\, dr + O\left(e^t\right) \quad \textit{as } t \to -\infty$$

and

$$Y(t) = \frac{t}{\sqrt{2}} \int_0^{+\infty} h(r) w'(r) e^w\, dr - \int_0^{+\infty} \left(\frac{2}{1 - e^{\sqrt{2}s}} + \frac{s}{\sqrt{2}} \right) h(s) w'(s) e^w\, ds + O\left(e^{-t}\right)$$
$$Y'(t) = \frac{1}{\sqrt{2}} \int_0^{+\infty} h(r) w'(r) e^w\, dr + O\left(e^{-t}\right) \quad \textit{as } t \to +\infty.$$

As we have done for the function w_ϵ, we have to add the projection of the function v_ϵ, namely the function β_ϵ given in the next lemma.

Lemma 3.4. (i) *The Cauchy problem:*

$$\begin{cases} -\beta''_\epsilon - \dfrac{(N-1)}{r} \beta'_\epsilon = \dfrac{(N-1)}{r} v'_\epsilon(r) \\ \\ \beta_\epsilon(r_0) = \beta'_\epsilon(r_0) = 0. \end{cases} \tag{3.32}$$

has the solution

$$\beta_\epsilon(r) = -(N-1) \int_r^{r_0} \frac{1}{t^{N-1}} \int_t^{r_0} \tau^{N-2} v'_\epsilon(\tau)\, d\tau\, dt.$$

(ii) *The following expansion holds*

$$\beta_\epsilon(\epsilon s + r_0) = \epsilon^2 \beta_1(s) + O\left(\epsilon^3 s^3\right), \quad \beta_1(s) := -\frac{(N-1)}{r_0} \int_0^s \int_0^\sigma v'(\rho)\, d\rho\, d\sigma. \tag{3.33}$$

(iii) *For any $r \in (0, r_0 - \delta)$*

$$\beta_\epsilon(r) = -\frac{(N-1)\nu_1}{r_0} \frac{(r - r_0)^2}{2} + O((r - r_0)^3) \tag{3.34}$$

Proof We argue as in Lemma 3.1. □

Unfortunately, the function $w_{\epsilon,r_0}(r)-\ln\lambda+\alpha_\epsilon(r)+v_\epsilon(r)+\beta_\epsilon(r)$ is yet a bad approximation of the solution near the boundary point r_0. We have to add an extra correction term z_ϵ given in next lemma, which solves a linear problem and *kills* all the ϵ^2-order terms (in particular, those in (3.19) and in (3.33)).

Lemma 3.5. (i) *There exists a solution z of the linear problem (see* (3.20), (3.21), (3.33), (3.26)*)*

$$-z''-e^w z=e^z\left[\alpha_2(s)+\beta_1(s)+\frac{1}{2}\left(\alpha_1(s)+v(s)\right)^2\right]\quad \text{in } \mathbb{R} \tag{3.35}$$

such that

$$z(s)=\zeta_1 s+\zeta_2+O\left(e^s\right)\quad \text{and}\quad z'(s)=\zeta_1+O\left(e^s\right)\quad \text{as } s\to-\infty$$

where $\zeta_1,\zeta_2\in\mathbb{R}$.

(ii) *In particular, the function $z_\epsilon(r):=\epsilon^2 z\left(\frac{r-r_0}{\epsilon}\right)$ is a solution of the linear problem*

$$-z_\epsilon''-e^{w_\epsilon}z_\epsilon=\epsilon^2 e^{w_\epsilon}\left\{\alpha_2\left(\frac{r-r_0}{\epsilon}\right)+\beta_1\left(\frac{r-r_0}{\epsilon}\right)+\frac{1}{2}\left[\alpha_1\left(\frac{r-r_0}{\epsilon}\right)+v\left(\frac{r-r_0}{\epsilon}\right)\right]^2\right\} \tag{3.36}$$

such that if $r\in[0,r_0-\delta]$ it satisfies

$$z_\epsilon(r)=\epsilon\zeta_1(r-r_0)+\zeta_2\epsilon^2+O\left(\epsilon^2 e^{-\frac{|r-r_0|}{\epsilon}}\right)\quad \text{as } \epsilon\to 0. \tag{3.37}$$

Proof

The result immediately follows by Lemma 3.3. □

4. The error estimate

Let us define the error term

$$\mathcal{R}_\lambda(\bar u_\lambda)=-\bar u_\lambda''-\frac{N-1}{r}\bar u_\lambda'+\bar u_\lambda-\lambda e^{\bar u_\lambda}. \tag{4.38}$$

where $\bar u_\lambda$ is defined as in (2.10).

First of all, it is necessary to choose constants a, b and c in (2.11) and A_1, A_2 and A_3 in (2.12) such that the approximate solutions in the neighborhood of the boundary and inside the interval glue up.

Lemma 4.1. *If*

$$a_1=A_1:=\frac{\sqrt{2}}{\mathcal{U}'(r_0)},\ a_2=A_2:=\frac{1}{\mathcal{U}'(r_0)}\left(\frac{\ln 4}{\mathcal{U}'(r_0)}-2\frac{N-1}{r_0}\right),\ a_3:=A_3-\nu_2,\ A_3:=\frac{\zeta}{\mathcal{U}'(r_0)} \tag{4.39}$$

then for any $r\in[r_0-2\delta,r_0-\delta]$ we have

$$u_1(r)-u_3(r)=O\left(e^{-\frac{|r-r_0|}{\epsilon}}\right)+O\left(\epsilon^2\right)+O\left(\epsilon(r-r_0)^2\right)+O\left((r-r_0)^3\right)+O\left(\frac{(r-r_0)^4}{\epsilon}\right),$$

$$u_1'(r)-u_3'(r)=O\left(\frac{1}{\epsilon}e^{-\frac{|r-r_0|}{\epsilon}}\right)+O\left(\epsilon\right)+O\left(\epsilon(r-r_0)\right)+O\left((r-r_0)^2\right)+O\left(\frac{(r-r_0)^3}{\epsilon}\right).$$

Proof Let us prove the first estimate. The proof of the second estimate is similar. By (2.11), by (3.22), (3.29), (3.34) and (3.37) we deduce that if $r \in [r_0 - 2\delta, r_0 - \delta]$ then

$$
\begin{aligned}
u_1(r) = \quad & \left[\ln\frac{4}{\epsilon^2} - \ln\lambda + \nu_2\epsilon\right] + \left[\frac{\sqrt{2}}{\epsilon} - \frac{(N-1)\ln 4}{r_0} + \nu_1 + \zeta_1\epsilon\right](r-r_0) \\
& + \left[\frac{(N-1)^2\ln 4}{r_0^2} - \frac{\sqrt{2}(N-1)}{r_0}\frac{1}{\epsilon} + \ln\frac{4}{\epsilon^2} - \ln\lambda - \frac{\nu_1(N-1)}{r_0}\right]\frac{(r-r_0)^2}{2} \\
& + \left[\frac{N(N-1)\sqrt{2}}{r_0^2}\frac{1}{\epsilon} + \sqrt{2}(N-1)\frac{1}{\epsilon} - \frac{N-1}{r_0}\left(\ln\frac{4}{\epsilon^2} - \ln\lambda\right)\right]\frac{(r-r_0)^3}{6} \\
& + O\left(e^{-\frac{|r-r_0|}{\epsilon}}\right) + O\left(\epsilon^2\right) + O\left(\frac{(r-r_0)^4}{\epsilon}\right) + O\left((r-r_0)^3\right) \\
& = \left[\frac{a_1}{\epsilon} + a_2 + a_3\epsilon + \nu_2\epsilon\right] + \left[\frac{\sqrt{2}}{\epsilon} - \frac{2(N-1)}{r_0} + a_1\sqrt{2}\ln 2 + \zeta_1\epsilon\right](r-r_0) \\
& + \left[-\frac{(N-1)\sqrt{2}}{r_0}\frac{1}{\epsilon} + \frac{a_1}{\epsilon} + a_2 + 2\frac{(N-1)^2}{r_0^2} - \frac{a_1(N-1)\sqrt{2}\ln 2}{r_0}\right]\frac{(r-r_0)^2}{2} \\
& + \left[\frac{N(N-1)\sqrt{2}}{r_0^2}\frac{1}{\epsilon} + \frac{\sqrt{2}}{\epsilon} - \frac{a_1(N-1)}{r_0}\frac{1}{\epsilon}\right]\frac{(r-r_0)^3}{6} \\
& + O\left(e^{-\frac{|r-r_0|}{\epsilon}}\right) + O\left(\epsilon^2\right) + O\left(\frac{(r-r_0)^4}{\epsilon}\right) + O\left((r-r_0)^3\right)
\end{aligned} \tag{4.40}
$$

On the other hand, by the mean value Theorem we deduce that

$$
\mathcal{U}(r) = \mathcal{U}(r_0) + \mathcal{U}'(r_0)(r-r_0) + \mathcal{U}''(r_0)\frac{(r-r_0)^2}{2} + \mathcal{U}''''(r_0)\frac{(r-r_0)^3}{6} + O\left((r-r_0)^4\right)
$$

with $\mathcal{U}(r_0) = 1$,

$$
\mathcal{U}''(r_0) = -\frac{N-1}{r_0}\mathcal{U}'(r_0) + \mathcal{U}(r_0) = -\frac{N-1}{r_0}\mathcal{U}'(r_0) + 1
$$

and

$$
\mathcal{U}'''(r_0) = -\frac{N-1}{r_0}\mathcal{U}''(r_0) + \frac{N-1}{r_0^2}\mathcal{U}'(r_0) + \mathcal{U}'(r_0) = \frac{N(N-1)}{r_0^2}\mathcal{U}'(r_0) + \mathcal{U}'(r_0) - \frac{N-1}{r_0}.
$$

These relations easily follow by differentiating (2.16). Therefore, if $r \in [r_0 - 2\delta, r_0 - \delta]$ we have

$$
\begin{aligned}
u_3(r) = \left(\frac{A_1}{\epsilon} + A_2 + A_3\epsilon\right)\mathcal{U}(r) = \quad & \left(\frac{A_1}{\epsilon} + A_2 + A_3\epsilon\right) + \left(\frac{A_1}{\epsilon} + A_2 + A_3\epsilon\right)\mathcal{U}'(r_0)(r-r_0) \\
& + \mathcal{U}''(r_0)\left(\frac{A_1}{\epsilon} + A_2\right)\frac{(r-r_0)^2}{2} + \mathcal{U}'''(r_0)\frac{A_1}{\epsilon}\frac{(r-r_0)^3}{6} \\
& + O\left(\epsilon(r-r_0)^2\right) + O\left((r-r_0)^3\right) + O\left(\frac{(r-r_0)^4}{\epsilon}\right)
\end{aligned} \tag{4.41}
$$

If (4.39) holds then combining (4.40) and (4.41) we easily get the claim.

□

Lemma 4.2. *There exists $C > 0$ and $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ we have*

$$
\|\mathcal{R}_\lambda\|_{L^1(0,r_0)} = O\left(\epsilon_\lambda^{1+\sigma}\right) \quad \textit{for some } \sigma > 0.
$$

Proof

Step 1. Evaluation of the error in $(r_0-\delta, r_0)$.

We use this estimate $1-e^t=-t-\frac{t^2}{2}+O(t^3)$ and we get

$$\begin{aligned}
\mathcal{R}_\lambda(u_1) &= -u_1''-\frac{N-1}{r}u_1'+u_1-\lambda e^{u_1}\\
&= -w_\epsilon''-\frac{N-1}{r_0}w_\epsilon'+w_\epsilon-\ln\lambda-\alpha_\epsilon''-\frac{N-1}{r}\alpha_\epsilon'\\
&\quad +\alpha_\epsilon-v_\epsilon''-\frac{N-1}{r}v_\epsilon'+v_\epsilon-\beta_\epsilon''-\frac{N-1}{r}\beta_\epsilon'+\beta_\epsilon\\
&\quad -z_\epsilon''-\frac{N-1}{r}z_\epsilon'+z_\epsilon-\lambda e^{w_{\epsilon,r_0}-\ln\lambda+\alpha_\epsilon+v_\epsilon+\beta_\epsilon+z_\epsilon}\\
&= \alpha_\epsilon+v_\epsilon+\beta_\epsilon+z_\epsilon-\frac{N-1}{r}z_\epsilon'\\
&\quad +e^{w_\epsilon}\left\{1-e^{\alpha_\epsilon+v_\epsilon+\beta_\epsilon+z_\epsilon}+v_\epsilon+z_\epsilon+\epsilon\alpha_1\left(\frac{r-r_0}{\epsilon}\right)\right.\\
&\quad \left.+\epsilon^2\left[\alpha_2\left(\frac{r-r_0}{\epsilon}\right)+\beta_1\left(\frac{r-r_0}{\epsilon}\right)+\frac12\left(\alpha_1\left(\frac{r-r_0}{\epsilon}\right)+v\left(\frac{r-r_0}{\epsilon}\right)\right)^2\right]\right\}\\
&= \alpha_\epsilon+v_\epsilon+\beta_\epsilon+z_\epsilon-\frac{N-1}{r}z_\epsilon'\\
&\quad +e^{w_\epsilon}\left\{-\alpha_\epsilon-\beta_\epsilon-\frac12\left(\alpha_\epsilon+v_\epsilon\right)^2+\epsilon\alpha_1\left(\frac{r-r_0}{\epsilon}\right)\right.\\
&\quad \left.+\epsilon^2\left[\alpha_2\left(\frac{r-r_0}{\epsilon}\right)+\beta_1\left(\frac{r-r_0}{\epsilon}\right)+\frac12\left(\alpha_1\left(\frac{r-r_0}{\epsilon}\right)+v\left(\frac{r-r_0}{\epsilon}\right)\right)^2\right]\right\}\\
&\quad +O\left(e^{w_\epsilon}|\alpha_\epsilon+v_\epsilon+\beta_\epsilon+z_\epsilon|^3\right)+O\left(e^{w_\epsilon}|\beta_\epsilon+z_\epsilon|^2\right)+O\left(e^{w_\epsilon}|\left(\alpha_\epsilon+v_\epsilon\right)\left(\beta_\epsilon+z_\epsilon\right)|\right)
\end{aligned}$$

because α_ϵ solves (3.18), v_ϵ solves (3.28), β_ϵ solves (3.32) and z_ϵ solves (3.36).

We have

$$\int\limits_{r_0-\delta}^{r_0}|\alpha_\epsilon+v_\epsilon+\beta_\epsilon+z_\epsilon|(r)dr=O\left(\int\limits_{r_0-\delta}^{r_0}\frac{(r-r_0)^2}{\epsilon}dr\right)=O\left(\frac{\delta^3}{\epsilon}\right)=O\left(\epsilon^{3\eta-1}\right),$$

because by (3.19), (3.33), the properties of v_ϵ in Lemma 3.2 and z_ϵ in 3.5 we deduce

$$\alpha_\epsilon(r)=O\left(\frac{(r-r_0)^2}{\epsilon}\right),\ \beta_\epsilon(r)=O\left((r-r_0)^2\right),\ v_\epsilon(r)=O\left(|r-r_0|+\epsilon\right),\ z_\epsilon(r)=O\left(\epsilon|r-r_0|+\epsilon^2\right).$$

By Lemma 3.5 we also deduce that $z_\epsilon'(r)=O(\epsilon)$ and so

$$\int\limits_{r_0-\delta}^{r_0}\left|\frac1r z_\epsilon'(r)\right|dr=O\left(\epsilon\delta\right)=O\left(\epsilon^{1+\eta}\right).$$

Moreover, we scale $s=\epsilon r+r_0$ and we get

$$\begin{aligned}&\int\limits_{r_0-\delta}^{r_0} e^{w_\epsilon}\left|-\alpha_\epsilon-\beta_\epsilon-\frac12\left(\alpha_\epsilon+v_\epsilon\right)^2+\epsilon\alpha_1\left(\frac{r-r_0}{\epsilon}\right)\right.\\&\left.+\epsilon^2\left[\alpha_2\left(\frac{r-r_0}{\epsilon}\right)+\beta_1\left(\frac{r-r_0}{\epsilon}\right)+\frac12\left(\alpha_1\left(\frac{r-r_0}{\epsilon}\right)+v\left(\frac{r-r_0}{\epsilon}\right)\right)^2\right]\right|dr\\=&\ \frac1\epsilon\int\limits_{-\delta/\epsilon}^{0} e^{w(s)}\left|-\alpha_\epsilon(\epsilon s+r_0)-\beta_\epsilon(\epsilon s+r_0)-\frac12\left(\alpha_\epsilon(\epsilon s+r_0)+\epsilon v(s)\right)^2+\epsilon\alpha_1(s)\right.\\&\left.+\epsilon^2\left[\alpha_2(s)+\beta_1(s)+\frac12\left(\alpha_1(s)+v(s)\right)^2\right]\right|ds\\=&\ O\left(\epsilon^2\int\limits_{\mathbb R}e^{w(s)}s^3\,ds\right)=O\left(\epsilon^2\right).\end{aligned}$$

Finally, we scale $s=\epsilon r+r_0$ and we get

$$\int\limits_{r_0-\delta}^{r_0} e^{w_\epsilon}|\alpha_\epsilon+v_\epsilon+\beta_\epsilon+z_\epsilon|^3dr=O\left(\int\limits_{r_0-\delta}^{r_0} e^{w_\epsilon}\left(|\alpha_\epsilon|^3+|v_\epsilon|^3+|\beta_\epsilon|^3+|z_\epsilon|^3\right)dr\right)$$

$$=O\left(\epsilon^2\int\limits_{\mathbb R}e^{w(s)}s^6ds+\epsilon^2\int\limits_{\mathbb R}e^{w(s)}v^3(s)ds+\epsilon^5\int\limits_{\mathbb R}e^{w(s)}s^6ds+\epsilon^5\int\limits_{\mathbb R}e^{w(s)}z^3(s)ds\right)=O\left(\epsilon^2\right)$$

$$\int\limits_{r_0-\delta}^{r_0} e^{w_\epsilon}|\beta_\epsilon+z_\epsilon|^2dr=O\left(\epsilon^3\int\limits_{\mathbb R}e^{w(s)}s^4\,ds+\epsilon^3\int\limits_{\mathbb R}e^{w(s)}z^2\,ds\right)=O\left(\epsilon^3\right)$$

$$\int\limits_{r_0-\delta}^{r_0} e^{w_\epsilon}|\left(\alpha_\epsilon+v_\epsilon\right)\left(\beta_\epsilon+z_\epsilon\right)|dr=O\left(\epsilon^2\int\limits_{\mathbb R}e^{w(s)}(s^2+|v|)(s^2+|z|)\,ds\right)=O\left(\epsilon^2\right),$$

because by (3.19) and (3.33) we deduce

$$\alpha_\epsilon(\epsilon s+r_0)=O\left(\epsilon s^2\right),\ \beta_\epsilon(\epsilon s+r_0)=O\left(\epsilon^2s^2\right).$$

By collecting all the previous estimates and taking into account the choice of η in (2.15) we get

$$\|\mathcal R_\lambda\|_{L^1(r_0-\delta,r_0)}=O\left(\epsilon^{1+\sigma}\right)\quad\text{for some }\sigma>0.\tag{4.42}$$

Step 2: Evaluation of the error in $(0,r_0-2\delta)$.

First of all, if δ is small enough (namely ϵ is small enough) we have

$$\mathcal U(r)\le\mathcal U(r_0-2\delta)=\mathcal U(r_0)+\mathcal U'(r_0)(-2\delta)+\frac12\mathcal U''(r_0-2\theta\delta)(2\delta)^2\le1-2\mathcal U'(r_0)\delta.$$

because $\mathcal U$ is increasing (see Lemma 2.1) and the mean value theorem applies for some $\theta\in(0,1)$.
Therefore, by (2.11), (2.15) and (4.39), we get

$$\mathcal R_\lambda(u_3)=-u_3''-\frac{N-1}{r}u_3'+u_3-\lambda e^{u_3}=-\lambda e^{\left(\frac{A_1}{\epsilon}+A_2+A_3\epsilon\right)\mathcal U(r)}=-\frac{4}{\epsilon^2}e^{(A_3-a_3)\epsilon}e^{\left(\frac{A_1}{\epsilon}+A_2+A_3\epsilon\right)[\mathcal U(r)-1]}$$
$$=O\left(\frac1{\epsilon^2}e^{-2A_1\mathcal U'(r_0)\frac\delta\epsilon}\right)=O\left(\frac1{\epsilon^2}e^{-2\sqrt2\frac{1}{\epsilon^{1-\eta}}}\right).$$

This implies that

$$\|\mathcal R_\lambda(u_3)\|_{L^1(0,r_0-2\delta)}=O\left(\epsilon^{1+\sigma}\right)\quad\text{for any }\sigma>0.\tag{4.43}$$

Step 3: Evaluation of the error in $[r_0-2\delta, r_0-\delta]$
We recall that $u_2=\chi u_1+(1-\chi)u_3$ hence

$$\begin{aligned}\mathcal{R}_\lambda(u_2) &= \chi\left[-u_1''-\frac{N-1}{r}u_1'+u_1\right]+(1-\chi)\left[-u_3''-\frac{N-1}{r}u_3'+u_3\right]\\ &\quad -2\chi'\left(u_1'-u_3'\right)+\left[-\chi''-\frac{N-1}{r}\chi'+\chi\right](u_1-u_3)-\lambda e^{\chi(u_1-u_3)+u_3}\\ &= \chi\mathcal{R}_\lambda(u_1)+(1-\chi)\mathcal{R}_\lambda(u_3)-\lambda\chi e^{u_1}\left[e^{(\chi-1)(u_1-u_3)}-1\right]+\lambda(1-\chi)e^{u_3}\\ &\quad -2\chi'\left(u_1'-u_3'\right)+\left[-\chi''-\frac{N-1}{r}\chi'+\chi\right](u_1-u_3)\end{aligned}$$

By Lemma (4.1) we immediately get (taking into account the choice of η in (2.15))

$$\int_{r_0-2\delta}^{r_0-\delta}\left|\chi'(r)\left(u_1'(r)-u_3'(r)\right)\right|dr=O\left(\delta^2\right)=O\left(\epsilon^{1+\sigma}\right),$$

$$\int_{r_0-2\delta}^{r_0-\delta}\left|\left[-\chi''(r)-\frac{N-1}{r}\chi'(r)+\chi(r)\right](u_1(r)-u_3(r))\right|(r)dr=O\left(\delta^2\right)=O\left(\epsilon^{1+\sigma}\right),$$

and

$$\int_{r_0-2\delta}^{r_0-\delta}\left|\lambda\chi e^{u_1(r)}\left[e^{(\chi(r)-1)(u_1(r)-u_3(r))}-1\right]\right|dr=O\left(\int_{r_0-2\delta}^{r_0-\delta}\lambda e^{u_1(r)}|u_1(r)-u_3(r)|dr\right)=O\left(\lambda\epsilon^2\right),$$

because $e^t-1=O(t)$. Arguing exactly as in Step 1 one proves that

$$\int_{r_0-2\delta}^{r_0-\delta}\chi(r)\left|\mathcal{R}_\lambda(u_1)(r)\right|dr=O\left(\epsilon^{1+\sigma}\right)$$

and arguing exactly as in Step 2 one proves that

$$\int_{r_0-2\delta}^{r_0-\delta}(1-\chi(r))\left|\mathcal{R}_\lambda(u_3)(r)\right|dr=O\left(\epsilon^{1+\sigma}\right)\quad\text{and}\quad\int_{r_0-2\delta}^{r_0-\delta}\lambda\left(1-\chi(r)\right)e^{u_3}(r)dr=O\left(\epsilon^{1+\sigma}\right).$$

Collecting all the previous estimates, we get

$$\|\mathcal{R}_\lambda(u_2)\|_{L^1(r_0-2\delta,r_0-\delta)}=O\left(\epsilon^{1+\sigma}\right)\quad\text{for some }\sigma>0. \tag{4.44}$$

The claim follows by (4.42), (4.43) and (4.44).

□

Lemma 4.3. *It holds that*

$$\lambda\epsilon_\lambda^2e^{u_\lambda(\epsilon_\lambda s+r_0)}\to e^{w(s)}\quad C^0\text{–}\textit{uniformly on compact sets of } (-\infty,0]\ \textit{as } \lambda\to 0 \tag{4.45}$$

and

$$\lambda\epsilon_\lambda\int_0^{r_0}e^{u_\lambda(r)}dr\to\int_{\mathbb{R}}e^{w(s)}ds\qquad\textit{as } \lambda\to 0 \tag{4.46}$$

Proof Let $[a,b]\subset(-\infty,0]$. If λ is small enough then

$$u_\lambda(\epsilon_\lambda s+r_0)=u_1(\epsilon_\lambda s+r_0)\quad\text{for any } s\in[a,b].$$

On the other hand, by (3.19), (3.33), the properties of v_ϵ in Lemma 3.2 and z_ϵ in 3.5 we deduce

$$\alpha_\epsilon(\epsilon s+r_0)+\epsilon v(s)+\beta_\epsilon(\epsilon s+r_0)+\epsilon^2 z(s)=O\left(\epsilon^2\right)+O\left(\epsilon|s|+\epsilon\right)+O\left(\epsilon^2 s^2\right)+O\left(\epsilon^2|s|+\epsilon^2\right)$$

and so

$$u_1(\epsilon s+r_0)=w(s)+\ln\frac{1}{\epsilon^2}-\ln\lambda+O\left(\delta|s|+\delta\right).$$

Therefore,

$$\lambda\epsilon_\lambda^2 e^{u_\lambda(\epsilon_\lambda s+r_0)}=e^{w(s)+O(\delta|s|+\delta)} \tag{4.47}$$

and (4.45) follows, since $s\in[a,b]$.

Moreover, since $w(s)=\sqrt{2}s+O\left(e^{\sqrt{2}s}\right)$ as s goes to $-\infty$, we also deduce that if λ (and also δ) is small enough there exist $a,b>0$ such that

$$\lambda\epsilon^2 e^{u_1(\epsilon s+r_0)}\le be^{-a|s|}\quad\text{for any } s\in(-\infty,0]. \tag{4.48}$$

Now, we have (scaling $r=\epsilon s+r_0$ in the first integral and arguing as in Step 3 of Lemma 4.2 to estimate the second and the third integral)

$$\begin{aligned}\lambda\epsilon_\lambda\int_0^{r_0} e^{u_\lambda(r)}dr &= \lambda\epsilon_\lambda\int_{r_0-\delta}^{r_0} e^{u_1(r)}dr+\lambda\epsilon_\lambda\int_{r_0-2\delta}^{r_0-\delta} e^{u_2(r)}dr+\lambda\epsilon_\lambda\int_0^{r_0-2\delta} e^{u_3(r)}dr\\ &= \lambda\epsilon_\lambda^2\int_{-\delta/\epsilon}^{0} e^{u_1(\epsilon_\lambda s+r_0)}dr+O\left(\epsilon^{1+\sigma}\right)\to\int_{\mathbb{R}} e^{w(s)}ds\quad\text{as }\lambda\to0,\end{aligned}$$

because of (4.47), (4.48) and dominate convergence Lebesgue's Theorem. That proves (4.46).

□

5. A contraction mapping argument and the proof of the main theorem

First of all we point out that $u_\lambda+\phi_\lambda$ is a solution to (2.9) if and only if ϕ_λ is a solution of the problem

$$\begin{cases}\mathcal{L}_\lambda(\phi_\lambda)=\mathcal{N}_\lambda(\phi_\lambda)+\mathcal{R}_\lambda(u_\lambda) & \text{in } (0,r_0)\\ \phi_\lambda'(0)=\phi_\lambda'(r_0)=0\end{cases} \tag{5.49}$$

where $R_\lambda(u_\lambda)$ is given in (4.38),

$$\mathcal{L}_\lambda(\phi_\lambda):=-\phi_\lambda''-\frac{N-1}{r}\phi_\lambda'+\phi_\lambda-\lambda e^{u_\lambda}\phi_\lambda$$

and

$$\mathcal{N}_\lambda(\phi_\lambda):=\lambda e^{u_\lambda+\phi_\lambda}-\lambda e^{u_\lambda}-\lambda e^{u_\lambda}\phi_\lambda.$$

The next result state that the linearized operator $\mathcal{L}_\lambda$ is uniformly invertible.

Proposition 5.1. *There exists $\lambda_0>0$ and $C>0$ such that for any $\lambda\in(0,\lambda_0)$ and for any $h\in L^\infty((0,r_0))$ there exists a unique $\phi\in W^{2,2}((0,r_0))$ solution of*

$$\begin{cases}\mathcal{L}_\lambda(\phi)=h\\ \phi'(0)=\phi'(r_0)=0\end{cases} \tag{5.50}$$

which satisfies

$$\|\phi\|_{L^\infty(0,r_0)}\le C\|h\|_{L^1(0,r_0)}$$

Proof By contradiction we assume that there exist sequences $\lambda_n \to 0$, $h_n \in L^\infty((0,r_0))$ and $\phi_n \in W^{2,2}((0,r_0))$ solutions of

$$\begin{cases} -\phi_n'' - \frac{N-1}{r}\phi_n' + \phi_n - \lambda_n e^{u_{\lambda_n}}\phi_n = h_n & \text{in } (0,r_0) \\ \phi_n'(0) = \phi_n'(r_0) = 0 \end{cases} \tag{5.51}$$

and

$$\|\phi_n\|_{L^\infty} = 1 \qquad \|h_n\|_{L^1} \to 0. \tag{5.52}$$

Let $\psi_n(s) = \phi_n(\epsilon_n s + r_0)$. Then ψ_n solves

$$\begin{cases} -\psi_n'' - \frac{N-1}{\epsilon_n s + r_0}\epsilon_n \psi_n' + \epsilon_n^2\psi_n - \lambda_n \epsilon_n^2 e^{u_n(\epsilon_n s + r_0)}\psi_n = \epsilon_n^2 h_n(\epsilon_n s + r_0) & \text{in } \left(-\frac{r_0}{\epsilon_n}, 0\right) \\ \psi_n'(-\frac{r_0}{\epsilon_n}) = \psi_n'(0) = 0 \end{cases} \tag{5.53}$$

We point out that, since ψ_n is bounded in $L^\infty((0,r_0))$, we get that, by standard elliptic regularity theory, $\psi_n \to \psi$ C^2- uniformly on compact sets of $(-\infty,0]$.
Hence we multiply the equation in (5.53) by a C_0^∞- test function, we integrate and we use (4.45) to deduce that ψ solves

$$\begin{cases} -\psi'' - e^w\psi = 0 & \text{in } (-\infty,0) \\ \|\psi\|_\infty \le 1 \\ \psi'(0) = 0. \end{cases} \tag{5.54}$$

A straightforward computation shows (see Lemma 4.2, [8]) that there exist $a, b \in \mathbb{R}$ such that

$$\psi(s) = a\frac{e^{\sqrt{2}s} - 1}{e^{\sqrt{2}s} + 1} + b\left(-2 + \sqrt{2}s\frac{e^{\sqrt{2}s} - 1}{e^{\sqrt{2}s} + 1}\right).$$

It is immediate to check that $b = 0$, since $\|\psi\|_\infty \le 1$ and also that $a = 0$, since $\psi'(0) = 0$. Therefore, $\psi \equiv 0$ in $(0,r_0)$.
We claim that $\|\phi_n\|_\infty = o(1)$. This immediately gives a contradiction since by assumption $\|\phi_n\|_\infty = 1$. To prove the claim we introduce the function G being the Green function of the operator $-u'' - \frac{N-1}{r}u' + u$ with Neumann boundary condition.
By (5.51), we deduce that

$$\begin{aligned} \phi_n(r) &= \int_0^{r_0} G(r,t)\lambda_n e^{u_{\lambda_n}}\phi_n(t)\,dt + \int_0^{r_0} G(r,t)h_n(t)\,dt \\ &= \epsilon_n\lambda_n \int_{-\frac{r_0}{\epsilon_n}}^0 G(r,\epsilon_n s + r_0)e^{u_{\lambda_n}(\epsilon_n s + r_0)}\psi_n(s)\,ds + \int_0^{r_0} G(r,t)h_n(t)\,dt \\ &= G(r)\epsilon_n\lambda_n \int_{-\frac{r_0}{\epsilon_n}}^0 e^{u_{\lambda_n}(\epsilon_n s + r_0)}\psi_n(s)\,ds + \int_0^{r_0} G(r,t)h_n(t)\,dt \\ &\quad + \epsilon_n\lambda_n \int_{-\frac{r_0}{\epsilon_n}}^0 \left[G(r,\epsilon_n s + r_0) - G(r)\right] e^{u_{\lambda_n}(\epsilon_n s + r_0)}\psi_n(s)\,ds \end{aligned}$$

Since G is bounded, it is immediate to check that $\int_0^{r_0} G(r,t)h_n(t)\,dt = o(1)$. We want to show that also

$$\epsilon_n\lambda_n \int_{-\frac{r_0}{\epsilon_n}}^0 \left[G(r,\epsilon_n s + r_0) - G(r)\right] e^{u_{\lambda_n}(\epsilon_n s + r_0)}\psi_n(s)\,ds = o(1) \tag{5.55}$$

If this is true then

$$\phi_n(r) = G(r)K_n + o(1)$$

where

$$K_n := \epsilon_n\lambda_n \int_{-\frac{r_0}{\epsilon_n}}^0 e^{u_{\lambda_n}(\epsilon_n s + r_0)}\psi_n(s)\,ds$$

We compute

$$G(r_0)K_n + o(1) = \phi_n(r_0) = \psi_n(0) = o(1)$$

and hence $K_n = o(1)$ since $G(r_0) \neq 0$. Then $\|\phi_n\|_\infty = o(1)$ and this gives a contradiction.
It remains to prove (5.55). We have:

$$\left|\epsilon_n\lambda_n \int_{-\frac{r_0}{\epsilon_n}} \left[G(r,\epsilon_n s + r_0) - G(r)\right] e^{u_{\lambda_n}(\epsilon_n s + r_0)}\psi_n(s)\, ds\right| \le \epsilon_n^2\lambda_n \int_{-\frac{r_0}{\epsilon_n}}^0 |s| e^{u_{\lambda_n}(\epsilon_n s + r_0)}|\psi_n(s)|\, ds$$

$$= \underbrace{\epsilon_n^2\lambda_n \int_{-\frac{\delta_n}{\epsilon_n}}^0 |s|e^{u_{1n}(\epsilon_n s + r_0)}|\psi_n(s)|\, ds}_{(I)} + + \underbrace{\epsilon_n^2\lambda_n \int_{-\frac{2\delta_n}{\epsilon_n}}^{-\frac{\delta_n}{\epsilon_n}} |s|e^{u_{2n}(\epsilon_n s + r_0)}|\psi_n(s)|\, ds}_{(II)}$$

$$+ \underbrace{\epsilon_n^2\lambda_n \int_{-\frac{r_0}{\epsilon_n}}^{-\frac{2\delta_n}{\epsilon_n}} |s|e^{u_{3n}(\epsilon_n s + r_0)}|\psi_n(s)|\, ds}_{(III)} = o(1)$$

Indeed, taking into account that $\psi_n \to 0$ pointwise in $(-\infty, 0)$ and $\|\psi_n\|_\infty \le 1$, by (4.48) we deduce

$$(I) = O\left(\int_{-\infty}^0 |s|e^{-a|s|}|\psi_n(s)|\, ds\right) = o(1)$$

for some $a > 0$, and arguing as in Step 2 and in Step 3 of Lemma 4.2, we get respectively

$$(III) = O\left(\int_{-\frac{r_0}{\epsilon_n}}^{-\frac{2\delta_n}{\epsilon_n}} |s|e^{-|s|}|\psi_n(s)|\, ds\right) = O\left(\int_{-\infty}^0 |s|e^{-|s|}|\psi_n(s)|\, ds\right) = o(1)$$

$$(II) = O\left(\int_{-\frac{2\delta_n}{\epsilon_n}}^{-\frac{\delta_n}{\epsilon_n}} |s|e^{-|s|}|\psi_n(s)|\, ds\right) = O\left(\int_{-\infty}^0 |s|e^{-|s|}|\psi_n(s)|\, ds\right) = o(1).$$

□

Finally, we are in position to use a contraction mapping argument to prove Theorem 1.1.

Proof[Proof of Theorem 1.1] By Proposition 5.1, we deduce that the linear operator $\mathcal{L}_\lambda$ is uniformly invertible and so problem (5.49) can be rewritten as

$$\phi = \mathcal{T}_\lambda(\phi) := \mathcal{L}_\lambda^{-1}\left[\mathcal{R}_\lambda(\bar{u}_\lambda) + \mathcal{N}_\lambda(\phi)\right]. \tag{5.56}$$

For a given number $\rho > 0$ let us consider the closed set $A_\rho := \left\{\phi \in L^\infty((0, r_0)) \ : \ \|\phi\|_\infty \le \rho\epsilon_\lambda^{1+\sigma}\right\}$ where ϵ_λ is defined in (2.11) and $\sigma > 0$ is given in Lemma 4.2.

We will prove that if λ is small enough, then $\mathcal{T}_\lambda : A_\rho \to A_\rho$ is a contraction map.
First of all, by (4.46) we get

$$\|\mathcal{N}_\lambda(\phi)\|_{L^1} \le \|\lambda e^{u_\lambda}\|_{L^1}\|\phi\|^2_{L^\infty} \le \frac{C}{\epsilon_\lambda}\|\phi\|^2_{L^\infty} \quad \text{for any } \phi \in A_\rho$$

and also

$$\|\mathcal{N}_\lambda(\phi_1) - \mathcal{N}_\lambda(\phi_2)\|_{L^1} \le \frac{C}{\epsilon_\lambda}\left(\max_{i=1,2}\|\phi_i\|_{L^\infty}\right)\|\phi_1 - \phi_2\|_{L^\infty} \quad \text{for any } \phi_1, \phi_2 \in A_\rho$$

for some $C > 0$.
By Lemma 4.2 we deduce that for some $\rho > 0$

$$\|\mathcal{T}_\lambda(\phi)\|_{L^\infty} \le C\left(\|\mathcal{R}_\lambda(u_\lambda)\|_{L^1} + \|\mathcal{N}_\lambda(\phi)\|_{L^1}\right) \le \rho\epsilon_\lambda^{1+\sigma}$$

and so $\mathcal{T}_\lambda$ maps A_ρ into itself. Moreover

$$\|\mathcal{T}_\lambda(\phi_1) - \mathcal{T}_\lambda(\phi_2)\|_{L^\infty} \le C\|\mathcal{N}_\lambda(\phi_1) - \mathcal{N}_\lambda(\phi_2)\|_{L^1} \le C\epsilon_\lambda^\sigma \|\phi_1 - \phi_2\|_{L^\infty}$$

which proves that for λ small enough $\mathcal{T}_\lambda$ is a contraction mapping on A_ρ, for a suitable ρ.

Therefore, $\mathcal{T}_\lambda$ has a unique fixed point in A_ρ, namely there exists a unique solution $\phi = \phi_\lambda \in A_\rho$ of the equation (5.56) or equivalently there exists a unique solution $u_\lambda + \phi_\lambda$ of problem (2.9). Estimate (1.5) follows by the definition of u_λ which coincides with u_3 far away from r_0. Indeed if $[a,b]$ is a compact set in $(0, r_0)$, we get that for λ small enough

$$\epsilon_\lambda \left(u_\lambda(r) + \phi_\lambda(r)\right) = (A_1 + A_2\epsilon_\lambda + A_3\epsilon_\lambda^2)\mathcal{U}(r) + \epsilon_\lambda\phi_\lambda(r) \to \frac{\sqrt{2}}{\mathcal{U}'(r_0)}\mathcal{U}(r) \text{ as } \lambda \to 0,$$

because of (4.39) and the fact $\|\phi_\lambda\|_{L^\infty} \to 0$ as $\lambda \to 0$.
Finally, estimate (1.4) follows by (4.46), taking into account that $\|\phi_\lambda\|_{L^\infty} \to 0$ as $\lambda \to 0$.

□

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